Complex Analysis: Midterm Exam

Aletta Jacobshal 01, Monday 14 December 2015, 09:00 – 11:00
Exam duration: 2 hours

Instructions — read carefully before starting

- Do not forget to very clearly write your **full name** and **student number** on each answer sheet and on the envelope. Do not seal the ennvelope.
- 10 points are "free" for handing-in the assignment. There are 5 questions and the total number of points is 100. The exam grade is the total number of points divided by 10.
- Solutions should be complete and clearly present your reasoning.

Question 1 (20 points)

Consider the function

$$f(z) = \frac{y-1}{x^2+(y-1)^2} + i\,\frac{x}{x^2+(y-1)^2},$$

where $z = x + iy$.

(a) (10 points) Prove that $f(z)$ is analytic for $z \neq i$.

Solution

We write

$$f(z) = u(x,y) + iv(x,y) = \frac{y-1}{x^2+(y-1)^2} + i\,\frac{x}{x^2+(y-1)^2}.$$

Then we have

$$\frac{\partial u}{\partial x} = -\frac{2x(y-1)}{(x^2+(y-1)^2)^2}$$
$$\frac{\partial u}{\partial y} = \frac{x^2-(y-1)^2}{(x^2+(y-1)^2)^2}$$
$$\frac{\partial v}{\partial x} = \frac{-x^2+(y-1)^2}{(x^2+(y-1)^2)^2}$$
$$\frac{\partial v}{\partial y} = -\frac{2x(y-1)}{(x^2+(y-1)^2)^2}$$

Therefore the Cauchy-Riemann equations are satisfied for all $x+iy$ with $x^2+(y-1)^2 \neq 0$ which is equivalent to $x \neq 0$ and $y \neq 1$ or $z = x+iy \neq i$.

Moreover, all the first order partial derivatives of u and v exist and are continuous in $D = \mathbb{C} \setminus \{i\}$ so the conditions are satisfied for f to be differentiable at every $z \in D$.

Since D is open f is analytic in D.

(b) (10 points) Write $f(z)$ as a function of z.

Solution

The expression $x^2+(y-1)^2$ can be recognized as $|x+i(y-1)|^2 = |x+iy-i|^2 = |z-i|^2$. Therefore

$$f(z) = \frac{(y-1)+ix}{|z-i|^2} = \frac{i(x-i(y-1))}{(z-i)\overline{(z-i)}} = \frac{i\overline{(z-i)}}{(z-i)\overline{(z-i)}} = \frac{i}{z-i}.$$

Question 2 (20 points)

Consider the two smooth arcs γ_1 and γ_2 shown in Figure 1. The arc γ_1 is a straight line from the point $-i$ to the point $1+i$ and the arc γ_2 is a smooth curve from $1+i$ to 2 for which $y = (x-2)^2$.

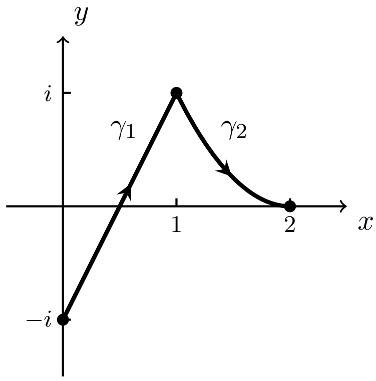

Figure 1: Smooth arcs γ_1 and γ_2 for Question 2.

(a) (10 points) Parameterize the two smooth arcs.

Solution

We denote by γ_1 the straight line from $-i$ to $1+i$. This can be parameterized by

$$z_1(t) = -i(1-t) + (1+i)t = t + i(2t-1), \quad 0 \le t \le 1.$$

We denote by γ_2 the curve from $1+i$ to 2. This is part of the graph of the function $y = (x-2)^2$ so it can be parameterized by

$$z_2(t) = t + i(t-2)^2, \quad 1 \le t \le 2.$$

(b) (10 points) Compute the integral

$$\int_{\gamma_1} \bar{z}^2 \, dz.$$

Solution

We have

$$\begin{aligned}
\int_{\gamma_1} \bar{z}^2 \, dz &= \int_0^1 \bar{z}_1(t)^2 z_1'(t) dt \\
&= \int_0^1 (t - i(2t-1))^2 (1+2i) dt \\
&= (1+2i) \int_0^1 (t^2 - (2t-1)^2 - 2it(2t-1)) dt \\
&= (1+2i) \int_0^1 [(-3-4i)t^2 + (4+2i)t - 1] dt \\
&= (1+2i) \left[\frac{1}{3}(-3-4i) + \frac{1}{2}(4+2i) - 1 \right] \\
&= -\frac{1}{3}(1+2i)i \\
&= \frac{1}{3}(2-i).
\end{aligned}$$

Question 3 (20 points)

Compute the value of the integral

$$\int_\Gamma \frac{\cos(\pi z)}{(z-i)(z-3)^2}\, dz$$

where Γ is the closed contour shown in Figure 2. Give the result as a complex number in Cartesian form $x + iy$.

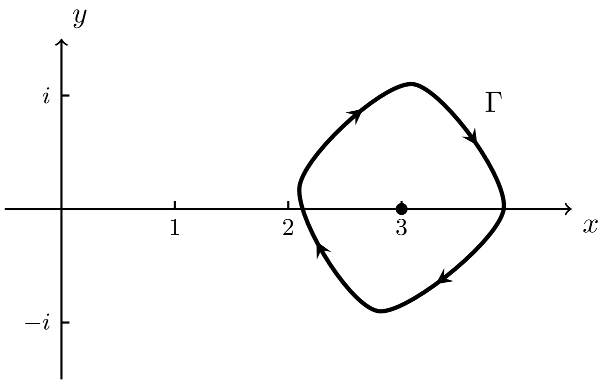

Figure 2: Contour Γ for Question 3.

Solution

The generalized Cauchy integral formula for $n = 1$ and $z_0 = 3$ gives that for a function $g(z)$ analytic on and inside Γ we have

$$\int_\Gamma \frac{g(z)}{(z-3)^2}\, dz = -2\pi i g'(3),$$

where the minus sign comes from the fact that Γ is negatively oriented.

Choosing

$$g(z) = \frac{\cos(\pi z)}{z-i},$$

we note that it is analytic on and inside Γ so it satisfies the conditions for applying the formula.

Therefore

$$\int_\Gamma \frac{\cos(\pi z)}{(z-i)(z-3)^2}\, dz = -2\pi i g'(3).$$

We then compute

$$g'(z) = -\frac{\cos(\pi z) + \pi(z-i)\sin(\pi z)}{(z-i)^2},$$

so

$$g'(3) = -\frac{\cos(3\pi) + \pi(3-i)\sin(3\pi)}{(3-i)^2} = \left(\frac{3+i}{10}\right)^2 = \frac{4+3i}{50}.$$

Finally,

$$\int_\Gamma \frac{\cos(\pi z)}{(z-i)(z-3)^2}\, dz = \frac{\pi}{25}(3-4i).$$

Question 4 (20 points)

Consider the function

$$f(z) = e^{\frac{1}{2}\,\mathrm{Log}(1-z)}\, e^{\frac{1}{2}\mathcal{L}_0(1+z)},$$

where $\mathcal{L}_0(z)$ is the branch of the logarithm function with a branch cut along the positive real axis $[0,\infty)$ and $\mathrm{Log}(z)$ is the principal value of the logarithm. Show that $f(z)$ is continuous at z in the interval $(1,\infty)$ on the real axis.

Solution

First, some remarks. Let $z \in (1,\infty)$. Then $1 - z < 0$ and the principal value of the logarithm is discontinuous on the negative real axis so $\mathrm{Log}(1-z)$ is discontinuous for $z \in (1,\infty)$. We also have $1 + z > 2$ and $\mathcal{L}_0$ is discontinuous along the positive real axis so $\mathcal{L}_0(1+z)$ is discontinuous for $z \in (-1,\infty)$ which means it is discontinuous for $z \in (1,\infty)$. These are indications that the function $f(z)$ could be discontinuous for $z \in (1,\infty)$. Nevertheless, these arguments are not a proof and we will instead show that the function is continuous for $z \in (1,\infty)$. Essentially, the two discontinuities cancel out. Moreover, it is possible to show that $f(z)$ is a branch of the multivalued function $(1-z^2)^{1/2}$ with branch cut the interval $(-1,1)$. I give below two possible solutions.

Proof 1

According to the remarks above the given function could be discontinuous along the interval $(1,\infty)$. Therefore, any possible discontinuity will manifest itself if we approach the interval from above and from below.

For this reason, let $z = x \pm i\varepsilon$ with $x > 1$ and $\varepsilon > 0$. Since we plan to take the limit $\varepsilon \to 0$ we can assume that ε is arbitrarily small. We have

$$1 - z = 1 - x \mp i\varepsilon$$

so

$$\mathrm{Log}(1-z) = \mathrm{Log}(1 - x \mp i\varepsilon).$$

Since $x > 1$ we have $1 - x < 0$ and therefore

$$\lim_{\varepsilon\to 0} \mathrm{Log}(1 - (x + i\varepsilon)) = \mathrm{Log}\,|1-x| - \pi i,$$

and

$$\lim_{\varepsilon\to 0} \mathrm{Log}(1 - (x - i\varepsilon)) = \mathrm{Log}\,|1-x| + \pi i.$$

We can write these limits together as

$$\lim_{\varepsilon\to 0} \mathrm{Log}(1 - (x \pm i\varepsilon)) = \mathrm{Log}\,|1-x| \mp \pi i,$$

and by the continuity of the exponential

$$\lim_{\varepsilon\to 0} e^{\frac{1}{2}\,\mathrm{Log}(1-(x\pm i\varepsilon))} = e^{\frac{1}{2}\lim_{\varepsilon\to 0}\mathrm{Log}(1-(x\pm i\varepsilon))} = e^{\frac{1}{2}\,\mathrm{Log}\,|1-x|} e^{\mp\frac{1}{2}\pi i} = \mp i\sqrt{x-1}.$$

We also have

$$1 + z = 1 + x \pm i\varepsilon$$

so

$$\mathcal{L}_0(1+z) = \mathcal{L}_0(1+x \pm i\varepsilon).$$

Since $x > 1$ we have $1 + x > 2$ and therefore

$$\lim_{\varepsilon\to 0} \mathcal{L}_0(1+x+i\varepsilon) = \operatorname{Log}(1+x),$$

and

$$\lim_{\varepsilon\to 0} \mathcal{L}_0(1+x-i\varepsilon) = \operatorname{Log}(1+x) + 2\pi.$$

By the continuity of the exponential

$$\lim_{\varepsilon\to 0} e^{\frac{1}{2}\mathcal{L}_0(1+x+i\varepsilon)} = e^{\frac{1}{2}\operatorname{Log}(1+x)} = \sqrt{1+x},$$

and

$$\lim_{\varepsilon\to 0} e^{\frac{1}{2}\mathcal{L}_0(1+x-i\varepsilon)} = e^{\frac{1}{2}[\operatorname{Log}(1+x)+2\pi i]} = e^{\frac{1}{2}\operatorname{Log}(1+x)} e^{i\pi} = -e^{\frac{1}{2}\operatorname{Log}(1+x)} = -\sqrt{1+x}.$$

Writing these limits together we have

$$\lim_{\varepsilon\to 0} e^{\frac{1}{2}\mathcal{L}_0(1+(x\pm i\varepsilon))} = \pm\sqrt{1+x}.$$

Finally, for $x > 1$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} f(x \pm i\varepsilon) &= \lim_{\varepsilon\to 0} e^{\frac{1}{2}\operatorname{Log}(1-(x\pm i\varepsilon))} \lim_{\varepsilon\to 0} e^{\frac{1}{2}\mathcal{L}_0(1+(x\pm i\varepsilon))} = (\mp i\sqrt{x-1})(\pm\sqrt{1+x}) \\ &= -i\sqrt{x-1}\sqrt{x+1} = -i\sqrt{x^2-1}.\end{aligned}$$

Since the two limits are equal the function is continuous.

Proof 2

Let $z = x + \varepsilon$ with $x > 1$ and $\varepsilon \in \mathbb{C}$. Since we plan to take the limit $\varepsilon \to 0$ we can assume that $|\varepsilon|$ is arbitrarily small. We have

$$1 - z = 1 - x - \varepsilon$$

so

$$\operatorname{Log}(1-z) = \operatorname{Log}(1-x-\varepsilon),$$

and

$$\lim_{\varepsilon\to 0} \operatorname{Log}(1-x-\varepsilon) = \lim_{\varepsilon\to 0} \operatorname{Log}|1-x-\varepsilon| + i \lim_{\varepsilon\to 0} \operatorname{Arg}(1-x-\varepsilon).$$

The absolute value is a continuous function and the real logarithm is continuous at positive real numbers so $\lim_{\varepsilon\to 0} \operatorname{Log}|1-x-\varepsilon| = \operatorname{Log}|1-x|$. Since $x > 1$ we have $1 - x < 0$ and therefore

$$\lim_{\varepsilon\to 0} \operatorname{Arg}(1-x-\varepsilon) = \begin{cases} -\pi, & \text{if } \operatorname{Im}\varepsilon > 0, \\ \pi, & \text{if } \operatorname{Im}\varepsilon < 0. \end{cases}$$

Therefore

$$\lim_{\varepsilon\to 0} \operatorname{Log}(1-x-\varepsilon) = \begin{cases} \operatorname{Log}|1-x| - \pi i, & \text{if } \operatorname{Im}\varepsilon > 0, \\ \operatorname{Log}|1-x| + \pi i, & \text{if } \operatorname{Im}\varepsilon < 0. \end{cases}$$

By the continuity of the exponential

$$\lim_{\varepsilon\to 0} e^{\frac{1}{2}\operatorname{Log}(1-x-\varepsilon)} = \begin{cases} e^{\frac{1}{2}\operatorname{Log}|1-x|}e^{-\pi i/2} = -i\sqrt{x-1}, & \text{if } \operatorname{Im}\varepsilon > 0, \\ e^{\frac{1}{2}\operatorname{Log}|1-x|}e^{\pi i/2} = i\sqrt{x-1}, & \text{if } \operatorname{Im}\varepsilon < 0. \end{cases}$$

We also have

$$1 + z = 1 + x + \varepsilon$$

so

$$\mathcal{L}_0(1+z) = \mathcal{L}_0(1+x+\varepsilon),$$

and

$$\lim_{\varepsilon\to 0} \mathcal{L}_0(1+x+\varepsilon) = \lim_{\varepsilon\to 0} \operatorname{Log}|1+x+\varepsilon| + i \lim_{\varepsilon\to 0} \arg_0(1+x+\varepsilon).$$

Since $x > 1$ we have $1 + x > 2$ and therefore

$$\lim_{\varepsilon\to 0} \operatorname{Log}|1+x+\varepsilon| = \operatorname{Log}(1+x),$$

and

$$\lim_{\varepsilon\to 0} \arg_0(1+x+\varepsilon) = \begin{cases} 0, & \text{if } \operatorname{Im}\varepsilon > 0, \\ 2\pi, & \text{if } \operatorname{Im}\varepsilon < 0. \end{cases}$$

Therefore

$$\lim_{\varepsilon\to 0} \mathcal{L}_0(1+x+\varepsilon) = \begin{cases} \operatorname{Log}(1+x), & \text{if } \operatorname{Im}\varepsilon > 0, \\ \operatorname{Log}(1+x) + 2\pi i, & \text{if } \operatorname{Im}\varepsilon < 0. \end{cases}$$

By the continuity of the exponential

$$\lim_{\varepsilon\to 0} e^{\frac{1}{2}\mathcal{L}_0(1+x+\varepsilon)} = \begin{cases} e^{\frac{1}{2}\operatorname{Log}(1+x)}e^{0} = \sqrt{x+1}, & \text{if } \operatorname{Im}\varepsilon > 0, \\ e^{\frac{1}{2}\operatorname{Log}(1+x)}e^{\pi i} = -\sqrt{x+1}, & \text{if } \operatorname{Im}\varepsilon < 0. \end{cases}$$

So, for $x > 1$, we have

$$\begin{aligned} \lim_{\varepsilon\to 0} f(x+\varepsilon) &= \lim_{\varepsilon\to 0} e^{\frac{1}{2}\operatorname{Log}(1-x-\varepsilon)} \lim_{\varepsilon\to 0} e^{\frac{1}{2}\mathcal{L}_0(1+x+\varepsilon)} \\ &= \begin{cases} (-i\sqrt{x-1})(\sqrt{x+1}), & \text{if } \operatorname{Im}\varepsilon > 0, \\ (i\sqrt{x-1})(-\sqrt{x+1}), & \text{if } \operatorname{Im}\varepsilon < 0, \end{cases} \\ &= -i\sqrt{x^2-1}. \end{aligned}$$

Since the limit does not depend on the direction of approach (that is, the sign of $\operatorname{Im}\varepsilon$) the function is continuous.

Question 5 (10 points)

Consider an entire function $f(z)$ such that its absolute value is bounded below by a positive number, that is, $|f(z)| > M > 0$ for all $z \in \mathbb{C}$. Show that $f(z)$ is constant.

Solution

Since $|f(z)| > M$ we conclude that $f(z) \neq 0$ for all $z \in \mathbb{C}$. This means that $g(z) = 1/f(z)$ is entire.

Then $|g(z)| = 1/|f(z)| < 1/M$ for all $z \in \mathbb{C}$, that is, $g(z)$ is bounded.

Since $g(z)$ is entire and bounded it must be constant, so $g(z) = c$, $c \in \mathbb{C}$, and therefore $f(z) = 1/g(z)$ is constant. In particular, $f(z) = 1/c$ for all $z \in \mathbb{C}$.

End of the exam (Total: 90 points)